ABELIAN STATE-CLOSED SUBGROUPS OF AUTOMORPHISMS OF m-ARY TREES

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Abstract. The group $\mathcal{A}_m$ of automophisms of a one-rooted m-ary tree admits a diagonal monomorphism which we denote by x. Let A be an abelian state-closed (or self-similar) subgroup of $\mathcal{A}_m$. We prove that the combined diagonal and tree-topological closure A^* of A is additively a finitely presented $\mathbb{Z}_m[[x]]$-module where $\mathbb{Z}_m$ is the ring of m-adic integers. Moreover, if A^* is torsion-free then it is a finitely generated pro-m group. Furthermore, the group A splits over its torsion subgroup. We study in detail the case where A^* is additivley a cyclic $\mathbb{Z}_m[[x]]$-module and we show that when m is a prime number then A^* is conjugate by a tree automorphism to one of two specific types of groups.

1. Introduction

Automorphisms of one-rooted regular trees $\mathcal{T}(Y)$ indexed by finite sequences from a finite set Y of size $m \geq 2$, have a natural interpretation as automata on the alphabet Y, and with states which are again automorphisms of the tree. A subgroup of the group of automorphisms $\mathcal{A}(Y)$ of the tree is said to be *state-closed,* in the language of automata (or *self-similar* in the language of dynamics) of degree m provided the states of its elements are themselves elements of the same group. If the group is not state-closed then we may consider its state-closure. The prime example of a state-closed group is the group generated by the binary adding machine $\tau = (e, \tau)\sigma$ where σ is the transposition $(0, 1)$.

We study in this paper representations of general abelian groups as state-closed groups of degree m. For this purpose we use topological and diagonal closure operations in the automorphism group of the tree. Representations of free abelian groups of finite rank as state-closed groups of degree 2 were characterized in [2].

An automorphism group G of the tree group is said to be *transitive* provided the permutation group $P(G)$ induced by G on the set Y is transitive; actions of groups on sets, will be applied on the right. It will be shown that the structure of state-closed groups can in a certain sense be reduced to those which are transitive.

The automorphism group $\mathcal{A}(Y)$ of the tree is a topological group with respect to the topology inherited from the tree. This topology allows us to exponentiate elements of $\mathcal{A}(Y)$ by m-ary integers from $\mathbb{Z}_m$. Given a subgroup G of $\mathcal{A}(Y)$, its

Date: October 02, 2009.

2000 *Mathematics Subject Classification.* Primary 20E08, 20F18.

Key words and phrases. Automorphisms of trees, state-closed groups, self-similar groups, abelian groups, topological closure, p-adic integers, pro-p groups.

The first author is grateful for the hospitality of the Department of Mathematics of the University of Brasília as well as travel assistance from the University of Wisconsin-Parkside. The second author acknowledges support from the Brazilian scientific agencies CNPq and FAPDF. Both authors are thankful to Laurent Bartholdi for making numerous comments.

topological closure $\overline{G}$ with respect to the tree topology belongs to the same variety as G. Also, if G is state-closed then so is $\overline{G}$.

The diagonal map $\alpha \to \alpha^{(1)} = (\alpha, \alpha, ..., \alpha)$ is a monomorphism of $\mathcal{A}_m$. Define inductively $\alpha^{(0)} = \alpha, \alpha^{(i+1)} = \left(\alpha^{(i)}\right)^{(1)}$ for $i \geq 0$. It is convenient to introduce a symbol x and write $\alpha^{(i)}$ as α^{x^i} for $i \geq 0$. This will permit more general exponentiation, by formal power series $p(x) \in \mathbb{Z}_m[[x]]$. Given a subgroup G of $\mathcal{A}(Y)$, its *diagonal closure* is the group $\widetilde{G} = \left\langle G^{(i)} \mid i \geq 0 \right\rangle$. Observe that the diagonal closure operation preserves the state-closed property.

We will show that given an abelian transitive state-closed group A, its diagonal closure $\widetilde{A}$ is again abelian. The composition of the diagonal and topological closures when applied to A produces an abelian group denoted by A^* which can be viewed additively as a finitely generated $\mathbb{Z}_m[[x]]$-module. This approach was first used in [1].

The prime decomposition $m = \prod_{1 \leq i \leq s} p_i^{k_i}$, provides us with the decomposition $\mathbb{Z}_m = \oplus_{1 \leq i \leq s} \varepsilon_i \mathbb{Z}_{p_i^{k_i}}$ where ε_i are orthogonal idempotents such that $1 = \sum_{1 \leq i \leq s} \varepsilon_i$, and provides us also with the decomposition $\mathbb{Z}_m[[x]] = \oplus_{1 \leq i \leq s} \varepsilon_i \mathbb{Z}_{p_i^{k_i}}[[x]]$. When $m = p^k$ and p a prime number, the rings $\mathbb{Z}_m[[x]]$ and $\mathbb{Z}_p[[x]]$ are isomorphic, yet when $k > 1$, they are different representations of the same object and for this reason we distinguish between them.

In Sections 3 and 4 we prove

Theorem 1. *Let A be an abelian transitive state-closed group of degree m. Then, (1) the group A^* is isomorphic to a finitely presented $\mathbb{Z}_m[[x]]$-module; (2) if A^* is torsion-free then it is a finitely generated $\mathbb{Z}_m$-module which is also a pro-m group.*

Item (1) is part of Theorem 5 and item (2) is Corollary 1 of Theorem 6.

We consider in Section 5 torsion subgroups of state-closed abelian groups and use methods from virtual endomorphisms of groups (see [3], [4]; reviewed in Subsection 5.1) to prove the following structural result.

Theorem 2. *Let A be an abelian transitive state-closed group of degree m and $tor(A)$ its torsion subgroup. Then, (i) $tor(A)$ is a direct summand of A and has exponent a divisor of the exponent of $P(A)$; (ii) the action of A on the m-ary tree induces transitive state-closed representations of $tor(A)$ on the m_1-tree and of $\frac{A}{tor(A)}$ on the m_2-tree, where $m_1 = |P(tor(A))|$ and $m_2 = |\frac{P(A)}{P(tor(A))}|$; (iii) if $A = tor(A)$ and $P(A) \cong \oplus_{1 \leq i \leq k} \frac{\mathbb{Z}}{m_i \mathbb{Z}}$, then $A^* \cong \oplus_{1 \leq i \leq k} \frac{\mathbb{Z}}{m_i \mathbb{Z}}[[x]]$.*

The above results are analogous to Theorem 4.3.4 of [5] on the structure of finitely generated pro-p groups. By item (i) of the theorem, an abelian torsion group G of infinite exponent cannot have a faithful representation as a transitive state-closed group for any finite degree. Put differently, the group G does not admit any simple virtual endomorphism. On the other hand, the group of automorphisms of the p-adic tree is replete with abelian p-subgroups of infinite exponent. Item (iii) follows from Theorem 7 which is a conjugacy result and therefore more general than isomorphism.

We focus our attention in Section 6 on transitive state-closed abelian groups A for which A^* is additively a cyclic $\mathbb{Z}_m[[x]]$-module. We show

Theorem 3. *(1) Let $q_{\mathbf{1}},...,q_{\mathbf{m}} \in \mathbb{Z}_m[[x]]$ and let σ be the cycle $(1,2,...,m)$. Then the expression*

$$\alpha = (\alpha^{q_{\mathbf{1}}},...,\alpha^{q_{\mathbf{m}}})\,\sigma$$

is a well-defined automorphism of the m-ary tree and the state-closure A of $\langle\alpha\rangle$ is an abelian transitive group. The group A^* is additively isomorphic to the quotient ring $\frac{\mathbb{Z}_m[[x]]}{(r)}$ where

$$r = m - xq \text{ and } q = q_{\mathbf{1}} + ... + q_{\mathbf{m}}.$$

(2) Let A be a transitive state-closed abelian group of degree m such that A^* is additively a cyclic $\mathbb{Z}_m[[x]]$-module. Then $P(A)$ is cyclic, say generated by σ, and A^* is the state-diagonal-topological closure of an element of the form $\alpha = (\alpha^{q_{\mathbf{1}}},...,\alpha^{q_{\mathbf{m}}})\,\sigma$ for some $q_{\mathbf{1}},...,q_{\mathbf{m}} \in \mathbb{Z}_m[[x]]$.

Finally, we provide a complete description of the group A^* for state-closed groups of prime degree. Let $j \geq 1$ and let $D_m(j)$ be the group generated by the set of states of the generalized adding machine $\alpha = \left(e,...,e,\alpha^{x^{j-1}}\right)\sigma$ acting on the m-ary tree with $\sigma = (1,2,...,m)$. The topological closure of $D_m(j)$ seen as $\mathbb{Z}_m$-module, is isomorphic to the ring $\frac{\mathbb{Z}_m[[x]]}{(r)}$, $r = m - x^j$.

Theorem 4. *Let A be an abelian transitive state-closed group of prime degree m and let σ be the m-cycle automorphism. If $tor(A)$ is nontrivial then A^* is a torsion group conjugate to $\langle\sigma\rangle^*$ $\left(\cong \frac{\mathbb{Z}}{m\mathbb{Z}}[[x]]\right)$. If A is torsion-free then A^* is a torsion-free group conjugate to the topological closure of $D_m(j)$ for some j.*

One of the questions which has remained unanswered is whether a free abelian group of infinite rank admits a faithful transitive state-closed representation, even of prime degree.

2. Preliminaries

We fix the notation $Y = \{1,2,...,m\}$, $\mathcal{T}_m = \mathcal{T}(Y)$, $\mathcal{A}_m = \mathcal{A}(Y)$ and we let $Perm(Y)$ be the group of permutations of Y. A permutation $\gamma \in Perm(Y)$ is extended to an automorphism of the tree by $\gamma : yu \to y^{\gamma}u$, fixing the non-initial letters of every sequence. An automorphism $\alpha \in \mathcal{A}_m$ is represented as $\alpha = (\alpha_1,\alpha_2,...,\alpha_m)\,\sigma(\alpha)$ where $\alpha_i \in \mathcal{A}_m$ and $\sigma(\alpha) \in Perm(Y)$. Successive developments of α_i produce for us α_u (a state of α) for every finite string u over Y.

The product of $\alpha = (\alpha_1,\alpha_2,...,\alpha_m)\,\sigma(\alpha)$ and $\beta = (\beta_1,\beta_2,...,\beta_m)\,\sigma(\beta)$ in $\mathcal{A}_m$, is

$$\alpha\beta = \left(\alpha_1\beta_{(1)\sigma(\alpha)},...,\alpha_m\beta_{(m)\sigma(\alpha)}\right)\sigma(\alpha)\,\sigma(\beta)\,.$$

Let G be a subgroup of $\mathcal{A}_m$. Denote the subgroup of G which fixes the vertices of the i-th level of the tree by $Stab_G(i)$. Given $y \in Y$, denote by $Fix_G(y)$ the subgroup of G consisting of the elements of G, which fix y. The group G is said to be *recurrent* provided it is transitive and $Fix_G(1)$ projects in the 1st coordinate onto G.

The group $\mathcal{A}_m$ is the inverse limit of its quotients by the i-th level stabilizers $Stab_{\mathcal{A}_m}(i)$ of the tree and is as such a topological group where each $Stab_{\mathcal{A}_m}(i)$ is an open and closed subgroup. For a subgroup G of automorphisms of the tree, its topological closure $\overline{G}$ coincides with the set of all infinite products $...g_{\mathbf{i}}...g_{\mathbf{1}}g_{\mathbf{0}}$, or alternately, $g_{\mathbf{0}}g_{\mathbf{1}}...g_{\mathbf{i}}...$ where $g_{\mathbf{i}} \in Stab_G(\mathbf{i})$. The group $\overline{G}$ satisfies the same group

identities as G. We note that the property of being state-closed is also preserved by the topological closure operation.

Let α be an automorphism of the tree. Then $\overline{\langle\alpha\rangle} = \{\alpha^p \mid p \in \mathbb{Z}_m\}$. More generally, for $q = \sum_{i\geq 0} q_i x^i \in \mathbb{Z}_m[[x]]$ with $q_i \in \mathbb{Z}_m$, we write the expression

$$\alpha^q = \alpha^{q_0}\alpha^{q_1 x}...\alpha^{q_i x^i}...$$

which can be verified to be a well-defined automorphism of the tree.

We recall the reduction of group actions to transitive ones, with a view to a similar reduction for state-closed groups of automorphisms of trees. Let G be a subgroup of $Perm(Y)$, let $\{Y_i \mid i = 1, ..., s\}$ be the set of orbits of G on Y and let $\{\rho_i : G \to Perm(Y_i) \mid i = 1, ..., s\}$ be the set of induced representations. Then, each ρ_i is transitive and $\rho : G \to \prod_{1\leq i\leq s} Perm(Y_i) \leq Perm(Y)$ defined by $g \to (g^{\rho_1}, ..., g^{\rho_s})$ is a monomorphism. The reduction for tree actions follows from

Lemma 1. *Let G be a state-closed group of automorphisms of the tree $\mathcal{T}(Y)$ and let X be a $P(G)$-invariant subset of Y. Then, $\mathcal{T}(X)$ is G-invariant and for the resulting representation $\mu : G \to \mathcal{A}(X)$, the group G^μ is state-closed. If G is diagonally closed or is topologically closed then so is G^μ.*

Proof. Let xu be a sequence from X and let $\alpha \in G$. Then, $(xu)^\alpha = x^{\sigma(\alpha)}u^{\alpha_x}$. As $x^{\sigma(\alpha)} \in X$ and $\alpha_x \in G$, it follows that $(xu)^\alpha$ is a sequence from X. Also, for any sequence u from X, we have $(\alpha^\mu)_u = (\alpha_u)^\mu$. Thus, G^μ is state-closed. The last assertion is clear. □

We note the following important properties of transitive state-closed abelian groups A.

Proposition 1. *Let A be an abelian transitive state-closed group of degree m. Then $Stab_A(i) \leq A^{(i)}$ for all $i \geq 0$. The group $\widetilde{A}$ is an abelian transitive state-closed group and is a minimal recurrent group containing A. Moreover, the topological closure and diagonal closure operations commute when applied to A. The diagonal-topological closure A^* of A is an abelian transitive state-closed group.*

Proof. Let $\alpha = (\alpha_1, ..., \alpha_m)\sigma, \beta = (\beta_1, ..., \beta_m) \in A$. Then, the conjugate of β by α is

$$\beta^\alpha = (\beta_1^{\alpha_1}, ..., \beta_m^{\alpha_m})^\sigma .$$

As $\alpha_i, \beta_i \in A$ and A is abelian, it follows that $\beta = (\beta_1, ..., \beta_m)^\sigma$. Furthermore, since A is transitive, $\beta = (\beta_1, ..., \beta_1) = (\beta_1)^{(1)}$. Thus, $Stab_A(i) \leq A^{(i)}$ for all i. A similar verification shows that $\widetilde{A} = \langle A^{(i)} \mid i \geq 0\rangle$ is abelian.

Let G be a recurrent group such that $A \leq G \leq \widetilde{A}$. Given $\alpha \in G$, as G is recurrent, there exists $\beta \in Stab_G(1)$ such that $\beta = (\beta_1, ..., \beta_m)$ with $\beta_1 = \alpha$. Since G is transitive and abelian, we have $\beta_1 = ... = \beta_m = \alpha$; that is, $\beta = \alpha^{(1)}$. Hence, $A^{(i)} \leq G$ and $G = \widetilde{A}$ follows.

The last two assertions of the proposition are clear. □

The following result indicates the smallness of recurrent transitive abelian groups, from the point of view of centralizers.

Proposition 2. *(Theorem 7 [4]) (1) Let A be a recurrent abelian group of degree m and let $C_{\mathcal{A}_m}(A)$ be the centralizer of A in $\mathcal{A}_m$. Then, $C_{\mathcal{A}_m}(A) = \overline{A}$. (2) Let m*

be a prime number and A be an infinite transitive state-closed abelian group. Then, $C_{\mathcal{A}_m}(A) = \overline{A}$.

This result will be used in the proofs of Lemma 3 and Step 4 of Theorem 9.

3. A presentation for A^*

Let A be a transitive abelian state-closed group of degree m and let A^* be its diagonal-topological closure. Then A^* is additively a $\mathbb{Z}_m[[x]]$-module having the following properties. Given $\alpha \in A^*$, then

$$\text{(i) } x\alpha = 0 \text{ implies } \alpha = 0; \text{(ii) } m\alpha = x\gamma \text{ for some } \gamma \in A^*.$$

Let $P(A)$ be given by its presentation

$$\langle \sigma_i \ (1 \leq i \leq k) \mid \sigma_i^{m_i} = e, \text{abelian} \rangle .$$

Choose for each σ_i an element β_i in A, which induces σ_i on Y; denote β_i by $\beta(\sigma_i)$. Then for any $n \geq 0$, the automorphism of the tree $\beta(\sigma_i)^{(n)}$ is an element of $\widetilde{A}$ which induces $(\sigma_i)^{(n)}$ on the $(n+1)$-th level of the tree. Although the notation β_i has been used to indicate the ith entry in an automorphism β, we hope this new usage will not cause confusion.

Theorem 5. *Let A be a transitive abelian state-closed group of degree m. Then A^* is additively a $\mathbb{Z}_m[[x]]$-module generated by*

$$\{\beta_i \mid 1 \leq i \leq k\}$$

subject to the set of defining relations

$$\left\{ r_i = \sum_{1 \leq j \leq k} m_i\beta_i - p_{ij}\beta_j x = 0 \mid 1 \leq i \leq k \right\} \text{ for some } p_{ij} \in \mathbb{Z}_m[[x]] .$$

Moreover, there exist $r, q \in \mathbb{Z}_m[[x]]$ such that $r = m - xq$ and $rA^* = (0)$. The elements of A^* can be represented additively as $\sum_{1\leq i\leq k} p_i\beta_i$ where $p_i = \sum_{j\geq 0} p_{ij}x^j$ and each $p_{ij} \in \mathbb{Z}$ with $0 \leq p_{ij} < m$.

Proof. Let $\alpha \in A^*$ and $\sigma(\alpha) = \prod_{1\leq i\leq k} \sigma_i^{r_{i1}}, 0 \leq r_{i1} < m_i$. Then, either $\alpha \left(\prod_{1\leq i\leq k} \beta_i^{r_{i1}}\right)^{-1}$ is the identity element or there exists $l_2 \geq 1$ such that

$$\alpha \left(\prod_{1\leq i\leq k} \beta_i^{r_{i1}}\right)^{-1} \in Stab(l_2) \backslash Stab(l_2+1)$$

and so, $\alpha \left(\prod_{1\leq i\leq k} \beta_i^{r_{i1}}\right)^{-1} = (\gamma)^{(l_2)}$ for some $\gamma \in A^*$. We treat γ in the same manner as α. In the limit, we obtain

$$\begin{aligned} \alpha &= \prod_{1\leq i\leq k} \left(\beta_i^{r_{i1}} (\beta_i^{r_{i2}})^{(l_2)} \ldots (\beta_i^{r_{ij}})^{(l_j)} \ldots\right) \\ &= \prod_{1\leq i\leq k} \beta_i^{q_i} \end{aligned}$$

where $0 \leq r_{ij} < m_i$, $1 \leq l_2 < l_3 < .. < l_j < ...$ and where $q_i = r_{i1} + \sum_{j \geq 2} r_{ij} x^{l_j}$ are formal power series in x. Additively then,

$$\alpha = \sum_{1 \leq i \leq k} q_i \beta_i \in \sum_{1 \leq i \leq k} \mathbb{Z}_{m_i} [[x]] \beta_i .$$

Each relation $\sigma_i^{m_i} = e$ in P produces in A^* a relation of the form

$$\beta_i^{m_i} = \prod_{1 \leq j \leq k} \beta_j^{x p_{ij}}$$

where p_{ij} are elements in the power series, as above; when written additively $\beta_i^{m_i}$ has the form

$$m_i \beta_i = x \left(\sum_{1 \leq j \leq k} p_{ij} \beta_j \right) .$$

Let $F = \oplus_{1 \leq i \leq k} \mathbb{Z}_m [[x]] \dot{\beta}_i$ be a free $\mathbb{Z}_m [[x]]$ module of rank k. Define the $\mathbb{Z}_m [[x]]$-homomorphism

$$\phi : \sum_{1 \leq i \leq k} \mathbb{Z}_m [[x]] \dot{\beta}_i \to A^*, \ \sum_{1 \leq i \leq k} p_i \dot{\beta}_i \to \prod_{1 \leq i \leq k} \beta_i^{p_i}$$

and let R be the kernel ϕ. Define J to be the $\mathbb{Z}_m [[x]]$-submodule of R generated by

$$\dot{r}_i = m_i \dot{\beta}_i - x \left(\sum_{1 \leq j \leq k} p_{ij} \dot{\beta}_j \right) \ (1 \leq i \leq k) .$$

We will show that $J = R$. So let $\nu \in R$ and write $\nu = \sum_{1 \leq i \leq k} \nu_i \dot{\beta}_i$ where

$$\begin{aligned} \nu_i &= \sum_{j \geq 0} \nu_{ij} x^j, \\ \nu_{ij} &= \nu_{ij,0} + m w_{ij} \in \mathbb{Z}_m . \end{aligned}$$

Then, $m_i | \nu_{i0,0}$, $\nu_{i0,0} = m_i \nu'_{i0,0}$; factor $m = m_i m'_i$. Therefore,

$$\begin{aligned} \nu_i &= \nu_{i0} + \left(\sum_{j \geq 1} \nu_{ij} x^{j-1} \right) x, \\ \nu_{i0} &= m_i \nu'_{i0,0} + m w_{i0} = \left(\nu'_{i0,0} + m'_i w_{i0} \right) m_i, \\ \nu_i \dot{\beta}_i &= \left(\nu'_{i0,0} + m'_i w_{i0} \right) \left(m_i \dot{\beta}_i \right) + \left(\sum_{j \geq 1} \nu_{ij} x^{j-1} \right) x \dot{\beta}_i, \\ &\equiv \left(\nu'_{i0,0} + m'_i w_{i0} \right) \left(x \sum_{1 \leq j \leq k} p_{ij} \dot{\beta}_j \right) + \left(\sum_{j \geq 1} \nu_{ij} x^{j-1} \right) x \dot{\beta}_i \text{ modulo } J. \end{aligned}$$

Hence,

$$\begin{aligned}\nu &= \sum_{1\leq i\leq k} \nu_i \dot{\beta}_i \in x\mu + J,\\ \mu &= \sum_{1\leq i\leq k} \mu_i \dot{\beta}_i \in R.\end{aligned}$$

Thence, by repeating the argument, we obtain

$$\begin{aligned}\nu &\in \left(\bigcap_{i\geq 1} x^i R\right) + J = J,\\ J &= R.\end{aligned}$$

On re-writing the relations $m_i\beta_i = \sum_{1\leq j\leq k} p_{ij} x \beta_j$ in the form

$$p_{i1}x\beta_1 + ... + (p_{ii}x - m_i)\,\beta_i + ... + p_{kk}x\beta_k = 0$$

we see that the $k \times k$ matrix of coefficients of these equations has determinant $r = m - xq$ for some $q \in \mathbb{Z}_m\,[[x]]$ and thus r annuls A^*.

The last assertion of the theorem follows by using $r = m - qx \in R$ to reduce the coefficients modulo m. □

4. The m-congruence property

A group G of automorphisms of the m-ary tree is said to satisfy the *m-congruence property* provided given m^i there exists $l\,(i) \geq 1$ such that $Stab_G(l\,(i)) \leq G^{m^i}$ for all i; in which case the topology on G inherited from $\mathcal{A}\,(Y)$ is equal to the pro-m topology. Since when A^* is written additively, we have $Stab_G(l\,(i)) = x^{l(i)}A^*$, the m-congruence property reads $x^{l(i)}A^* \leq m^i A^*$.

Theorem 6. *Let $r = m - qx^j \in \mathbb{Z}_m\,[[x]]$ with $q \in \mathbb{Z}_m\,[[x]]$ and $j \geq 1$. Let S be quotient ring $\frac{\mathbb{Z}_m[[x]]}{(r)}$. Suppose S is torsion-free. Then, S is a finitely generated pro-m group.*

Proof. From the decomposition $\mathbb{Z}_m\,[[x]] = \oplus_{1\leq i\leq s}\varepsilon_i \mathbb{Z}_{p_i^{k_i}}\,[[x]]$ corresponding to the prime decomposition $m = \prod_{1\leq i\leq s} p_i^{k_i}$, we obtain

$$\begin{aligned}r &= \sum_{1\leq i\leq s} r_i,\\ r_i &= \varepsilon_i r = p_i^{k_i} - q_i\,(x)\,x^j,\\ S &= \sum_{1\leq i\leq s} S_i, \quad S_i = \frac{\mathbb{Z}_{p_i^{k_i}}\,[[x]]}{(r_i)}\end{aligned}$$

where each S_i is torsion-free. Thus, it is sufficient to address the case where m is a prime power p^k.

(1) First, we show that S is a pro-m group.

So, let $r = p^k - qx^j$ and decompose $q = q\,(x) = s\,(x) + p.t\,(x)$ where each non-zero coefficient of $s\,(x)$ is an integer relatively prime to p. If $s\,(x) = 0$ then $q\,(x) = p.t\,(x)$ and

$$\begin{aligned}r &= p^k - q\,(x)\,x^j = p^k - p.t\,(x)\,x^j\\ &= p\left(p^{k-1} - t\,(x)\,x^j\right) \in (r)\,;\end{aligned}$$

but as by hypothesis S is torsion free, we have $p^{k-1} - t(x) x^j \in (r)$ which is not possible.

Write $s(x) = x^l u(x)$ where $l \geq 0$ and where $u(x)$ is invertible in $\mathbb{Z}_m[[x]]$ with inverse $u'(x)$. Then, $q(x) = x^l u(x) + p.t(x)$ and

$$\begin{aligned} r &= p^k - \left(x^l u(x) x^j + p.t(x) x^j\right) \\ &= p\left(p^{k-1} - t(x) x^j\right) - x^{j+l} u(x). \end{aligned}$$

Therefore, on multiplying by $u'(x)$, the inverse of $u(x)$, we obtain

$$p\left(p^{k-1} - t(x) x^j\right) u'(x) \equiv x^{j+l} \text{ modulo } r.$$

It follows that

$$x^{j+l} S \leq pS, \quad x^{n(j+l)} S \leq p^n S.$$

(2) Now we show that S is finitely generated as a $\mathbb{Z}_m$-module.

By the previous step there exist $l \geq 1$ and $v(x) \in \mathbb{Z}[[x]]$ such that

$$x^l \equiv mv(x) \mod r.$$

Decompose $v(x) = v_1(x) + v_2(x) x^l$ where the degree of $v_1(x)$ is less that l. Then, we deduce modulo r:

$$\begin{aligned} v(x) &\equiv v_1(x) + v_2(x) mv(x), \\ v_2(x) v(x) &\equiv w(x) \in \mathbb{Z}[[x]], \\ w(x) &= w_1(x) + w_2(x) x^l, \\ v(x) &\equiv v_1(x) + mw(x) \\ &\equiv v_1(x) + mw_1(x) + mw_2(x) x^l \\ &\dots \\ v(x) &\equiv a_0 + a_1 x + \dots + a_1 x^{l-1},\ a_i \in \mathbb{Z}_m. \end{aligned}$$

We have shown that S is generated by $1, x, \dots, x^{l-1}$ as a pro-m group. □

Corollary 1. *Let A be an abelian transitive state-closed group of degree m. Suppose the group A^* is torsion-free. Then A^* is a finitely generated pro-m group.*

Proof. With previous notation, the group A^* is a $\mathbb{Z}_m[[x]]$-module generated by

$$\{\beta_i = \beta(\sigma_i) \mid 1 \leq i \leq k\}$$

and is annihilated by $r = m - qx^j \in \mathbb{Z}_m[[x]]$ for some $q \in \mathbb{Z}_m[[x]]$ and $j \geq 1$.

It follows that A^* is an S-module where $S = \frac{\mathbb{Z}_m[[x]]}{(r)}$. Since S satisfies the m-congruence property, it follows that A^* is a pro-m group.

That A^* is a finitely generated $\mathbb{Z}_m$-module, is a consequence of S being a finitely generated $\mathbb{Z}_m$-module. □

5. Torsion in state-closed abelian groups

5.1. **Preliminaries on virtual endomorphisms of groups.** Let G be a transitive state-closed subgroup of $\mathcal{A}(Y)$ where $Y = \{1, 2, \dots, m\}$. Then $[G : Fix_G(1)] = m$ and the projection on the 1st coordinate of $Fix_G(1)$ produces a subgroup of G; that is, $\pi_1 : Fix_G(1) \rightarrow G$ is a virtual endomorphism of G. This notion has proven to be effective in studying state-closed groups. We give a quick review below.

Let G be a group with a subgroup H of finite index m and a homomorphism $f : H \to G$. A subgroup U of G is *semi-invariant* under the action of f provided $(U \cap H)^f \leq U$. If $U \leq H$ and $U^f \leq U$ then U is f-*invariant*.

The largest subgroup K of H, which is normal in G and is f-invariant is called the f-core(H). If the f-core(H) is trivial then f and the triple (G, H, f) are said to be a *simple*.

Given a triple (G, H, f) and a right transversal $L = \{x_1, x_2, ..., x_m\}$ of H in G, the permutational representation $\pi : G \to Perm\,(1, 2, ..., m)$ is $g^\pi : i \to j$ which is induced from the right multiplication $Hx_ig = Hx_j$. We produce recursively a representation $\varphi : G \to \mathcal{A}\,(m)$ as follows:

$$g^\varphi = \left(\left(x_i g.\left(x_{(i)g^\pi}\right)^{-1}\right)^{f\varphi}\right)_{1\leq i\leq m} g^\pi.$$

One further expansion of g^φ is

$$\begin{aligned} g^\varphi &= \left(\left(\left(x_j g_i . x^{-1}_{(j)g_i^\pi}\right)^{f\varphi}\right)_{1\leq j\leq m} g_i^\pi\right)_{1\leq i\leq m} g^\pi, \\ &= \left(\left(\left(x_j g_i . x^{-1}_{(j)g_i^\pi}\right)^{f\varphi}\right)_{1\leq j\leq m}\right)_{1\leq i\leq m} (g_i^\pi)_{1\leq i\leq m}\ g^\pi \end{aligned}$$

where $g_i = \left(x_i g . x^{-1}_{(i)g^\pi}\right)^f$

The kernel of φ is precisely the f-core(H), G^φ is state-closed and $H^\varphi = Fix_{G^\varphi}(1)$.

5.1.1. *Changing transversals.* We will show below that changing the transversal of H in G produces another representation of G, conjugate to the original one by an explicit automorphism of the m-ary tree.

Proposition 3. *Let (G, H, f) be a triple and*

$$L = \{x_1, x_2, ..., x_m\}\,, L' = \{x'_1 = h_1x_1, x'_2 = h_2x_2, ..., x'_m = h_mx_m\}$$

right transversals of H in G where $h_i \in H$. Let $\varphi = \varphi_{x_i}, \varphi' = \varphi_{h_ix_i} : G \to \mathcal{A}\,(m)$ be the corresponding tree representations and define the following elements of $\mathcal{A}\,(m)$,

$$\begin{aligned} \gamma &= \gamma_{h_i,\varphi'} = \left((h_i)^{f\varphi'}\right)_{1\leq i\leq m}, \\ \lambda &= \lambda_{h_i,\varphi'} = \gamma\gamma^{(1)}...\gamma^{(n)}.... \end{aligned}$$

Then,

$$\varphi_{h_ix_i} = \varphi_{x_i}\left(\lambda_{h_i^{-1},\varphi_{x_i}}\right).$$

Proof. The representations $\varphi, \varphi' : G \to \mathcal{A}\,(m)$ are defined by

$$\begin{aligned} g^\varphi &= \left(\left(x_i g.\left(x_{(i)g^\pi}\right)^{-1}\right)^{f\varphi}\right)_{1\leq i\leq m} g^\pi, \\ g^{\varphi'} &= \left(\left(x'_i g.\left(x'_{(i)g^\pi}\right)^{-1}\right)^{f\varphi'}\right)_{1\leq i\leq m} g^\pi. \end{aligned}$$

The relationship between φ' and φ is established as follows,

$$\begin{aligned} g^{\varphi'} &= \left(\left(h_i x_i g.\left(h_{(i)g^\pi} x_{(i)g^\pi}\right)^{-1}\right)^{f\varphi'}\right)_{1\leq i\leq m} g^\pi \\ &= \left(\left(h_i\left(x_i g.x^{-1}_{(i)g^\pi}\right)h^{-1}_{(i)g^\pi}\right)^{f\varphi'}\right)_{1\leq i\leq m} g^\pi \\ &= \left((h_i)^{f\varphi'}\right)_{1\leq i\leq m}.\left(\left(x_i g.x^{-1}_{(i)g^\pi}\right)^{f\varphi'}\right)_{1\leq i\leq m}.\left(\left(h^{-1}_{(i)g^\pi}\right)^{f\varphi'}\right)_{1\leq i\leq m} g^\pi \\ &= \left((h_i)^{f\varphi'}\right)_{1\leq i\leq m}.\left(\left(x_i g.x^{-1}_{(i)g^\pi}\right)^{f\varphi'}\right)_{1\leq i\leq m} g^\pi.\left((h_i)^{f\varphi'}\right)^{-1}_{1\leq i\leq m}. \end{aligned}$$

Therefore

$$g^{\varphi'} = \gamma.\left(\left(x_i g.x^{-1}_{(i)g^\pi}\right)^{f\varphi'}\right)_{1\leq i\leq m} g^\pi.\gamma^{-1}$$

where $\gamma = \left((h_i)^{f\varphi'}\right)_{1\leq i\leq m}$ is independent of g. Repeating this development for each $g_i = \left(x_i g.x^{-1}_{(i)g^\pi}\right)^f$, we find that

$$g^{\varphi'} = \gamma\gamma^{(1)}.\left(\left(\left(x_j g_i.x^{-1}_{(j)g_i^\pi}\right)^{f\varphi'}\right)_{1\leq j\leq m} g_i^\pi\right)_{1\leq i\leq m} g^\pi.\gamma^{-(1)}\gamma^{-1}.$$

Thus in the limit, we obtain $\lambda = \gamma\gamma^{(1)}...\gamma^{(n)}...$ such that

$$\begin{aligned} g^{\varphi'} &= \lambda g^{\varphi}\lambda^{-1} \text{ for all } g\in G, \\ \varphi &= \varphi'\lambda. \end{aligned}$$

Introducing the explicit dependence of $\varphi, \varphi', \lambda$ on the transversals, the previous equation becomes

$$\varphi_{x_i} = \left(\varphi_{h_i x_i}\right)\left(\lambda_{h_i,\varphi_{h_i x_i}}\right).$$

On replacing h_i by h_i^{-1} and on denoting $h_i^{-1}x_i$ by x_i', we obtain

$$\varphi_{h_i x_i'} = \left(\varphi_{x_i'}\right)\left(\lambda_{h_i^{-1},\varphi_{x_i'}}\right).$$

□

Example 1. *Let $G = C = \langle a\rangle$ be the infinite cyclic group, let $H = \langle a^2\rangle$ and let $f : H \to G$ be defined by $a^2 \to a$. Given $l, k \geq 0$, then on choosing the transversal $L_{k,l} = \{a^{2k}, a^{2l+1}\}$ for H in G, we obtain the representation $\varphi_{k,l} : G \to \mathcal{A}(m)$ where $\varphi_{k,l} : a \to \alpha = \left(\alpha^{k-l}, \alpha^{-k+l+1}\right)\sigma$.*

5.1.2. *Subtriples, Quotient triples.* Given a triple (G, H, f) and given subgroups $V \leq G, U \leq H \cap V$ such that $(U)^f \leq V$, we call $(V, U, f|_U)$ a *sub-triple* of G. If N is a normal semi-invariant subgroup of G then $\overline{f} : \frac{HN}{N} \to \frac{G}{N}$ given by $\overline{f} : Nh \to Nh^f$ is well-defined and $\left(\frac{G}{N}, \frac{HN}{N}, \overline{f}\right)$ is a *quotient triple*.

Let (G, H, f) be a simple triple where G is abelian and $[G : H] = m$. Then, any sub-triple of G is simple. Let $T = tor(G)$ denote the torsion subgroup of G and for $l \geq 1$ define $G(l) = \{g \in T \mid o(g)\,|l\}, H(l) = G(l) \cap H$. Then, clearly, $f : tor(H) \to tor(G)$ and $f : H(l) \to G(l)$. Therefore, $tor(G)$ and $G(l)$ are

semi-invariant and $\left(tor\left(G\right), tor\left(H\right), f|_{tor(H)}\right)$ and $\left(G\left(l\right), H\left(l\right), f|_{H(l)}\right)$ are simple *sub-triples.*

Lemma 2. *Let (G, H, f) be a simple triple. The triple $\left(\frac{G}{G(l)}, \frac{HG(l)}{G(l)}, \overline{f}\right)$ is also simple.*

Proof. For suppose $K \leq H$ is such that $G\left(l\right) K^{f} \leq G\left(l\right) K$. Then

$$\left(G\left(l\right) K^{f}\right)^{l} = \left(K^{f}\right)^{l} = \left(K^{l}\right)^{f} \leq \left(G\left(l\right) K\right)^{l} = \left(K\right)^{l};$$

that is, K^{l} is f-invariant. Since f is simple, $K^{l} = \{e\}$ and so, $K \leq G\left(l\right)$. □

5.2. **The torsion subgroup.**

Proposition 4. *Let A be transitive state-closed abelian group of degree m. Then $tor\left(A\right)$ has finite exponent and is therefore a direct summand of A.*

Proof. Let $T = tor\left(A\right)$, $A_1 = Stab_A\left(1\right)$, $T_1 = T \cap A_1$and $[T : T_1] = m'$. Then, the projection on the 1st coordinate of T_1 is a subgroup of T and the triple $(T, T_1, \pi_1|_{T_1})$ is simple of degree $m'|m$; let $m = m'm''$. Hence, in this representation, T is a torsion transitive state-closed subgroup of $\mathcal{A}_{m'}$, the automorphism group of the tree $\mathcal{T}_{m'}$.

Fixing this last representation of T, let $Q = P\left(T\right)$ and let σ_i $(1 \leq i \leq k)$ be a minimal set of generators of Q and as before, let $\beta_i = \beta\left(\sigma_i\right) \in T$ be such that $\sigma\left(\beta_i\right) = \sigma_i$. Let r be the maximum order of the elements $\beta_1, ..., \beta_k$. As any $\alpha \in T$ can be written in the form

$$\alpha = \prod_{1 \leq i \leq k} \beta_i^{r_{i1}} \left(\beta_i^{r_{i2}}\right)^{(l_2)} ... \left(\beta_i^{r_{ij}}\right)^{(l_j)} ...$$

it follows that $\alpha^{r} = e$.

Since T has finite exponent, it is a pure bounded subgroup of A and therefore it is a direct summand of A ([6], Th. 4.3.8). □

We recall a classic example of an abelian group G which does not split over its torsion subgroup (see [6], page 108).

Example 2. *Let G be the direct product of groups $\prod_{i \geq 1} C_i$ where $C_i = \langle c_i \rangle$ is cyclic of order p^i and let H be the direct sum $\sum_{i \geq 1} C_i$. Then $H \leq tor\left(G\right) = \bigcup_{l \geq 1} G\left(p^l\right)$. Moreover, H is a basic subgroup of G and in particular, $\frac{G}{H}$ is p-divisible. This observation leads directly to a proof that G does not split over $tor\left(G\right)$.*

The proof of the previous proposition did not establish the exponent of $tor\left(A\right)$. This we do in the next two lemmas.

Lemma 3. *Let m be a prime number and A an abelian transitive state-closed torsion group of degree m. Then, A is conjugate by a tree automorphism to a subgroup of the diagonal-topological closure of $\langle \sigma \rangle$ and so has exponent m.*

Proof. We observe that, $A\left(m\right)$ is not contained in $A_1 = Stab_A\left(1\right)$. For otherwise, $A\left(m\right)$ would be invariant under the projection on the 1st coordinate. Choose $a \in A \backslash A_1$ of order m. Therefore, $A = A_1 \langle a \rangle$. On choosing $\left\{a^i \mid 0 \leq i \leq m-1\right\}$ as a transversal of A_1 in A, the image of a acquires the form $\sigma = (1, ..., m)$ in this tree representation of A. Thus, we may suppose by Proposition 3 that $\sigma \in A$.

Therefore, $\widetilde{A}$ contains the subgroup $\widetilde{\langle\sigma\rangle} = \left\langle \sigma^{(i)} \mid i \geq 0 \right\rangle$. By Proposition 2, we have $C_{\mathcal{A}}\widetilde{\langle\sigma\rangle} = \langle\sigma\rangle^*$ and thus, $A \leq C_{\mathcal{A}}(A) \leq \langle\sigma\rangle^*$. □

Lemma 4. *Suppose A is an abelian transitive state-closed torsion group of degree m. Then the exponent of A is equal to the exponent of $P(A)$.*

Proof. By induction on $|P(A)| = m$. The exponent of A is a multiple of the exponent of $P(A)$. By the previous lemma, we may assume m to be composite. Let p be a prime divisor of m and $A(p) = \{a \in A \mid a^p = e\}$. Then, $A(p)$ is a nontrivial subgroup and $P(A(p)) \leq \{\sigma \in P \mid \sigma^p = e\}$. By Lemma 2, $\left(\frac{A}{A(p)}, \frac{A_1 A(p)}{A(p)}, \overline{\pi_0}\right)$ is simple; also, $P\left(\frac{A}{A(p)}\right) = \frac{P(A)}{P(A(p))}$. The proof follows by induction. □

Theorem 7. *Suppose A is an abelian transitive state-closed torsion group of degree m. Then, A is conjugate to a subgroup of the topological closure of*

$$\widetilde{P(A)} = \left\langle \sigma^{(i)} \mid \sigma \in P(A), \ i \geq 0 \right\rangle.$$

Proof. Let $P = P(A)$ have exponent r and let B be a maximal homogeneous subgroup of P of exponent r (that is, B is a direct sum of cyclic groups of order r), minimally generated by $\{\sigma_i \mid (1 \leq i \leq s)\}$. Choose for each σ_i an element $\beta_i = \beta(\sigma_i) \in A$ and let $\dot{B} = \langle \beta_i \mid (1 \leq i \leq s)\rangle$. Then, as the order of each β_i is a multiple of r, while the exponent of A is r, we conclude from the previous lemma that $o(\beta_i) = o(\sigma_i) = r$ for $(1 \leq i \leq s)$. Since $\beta_i \to \sigma_i$ defines a projection of $\dot{B}$ onto B we conclude that $\dot{B} \cong B$ and $\dot{B} \cap A_1 = \{e\}$, where $A_1 = Stab_A(1)$.

Clearly $\dot{B}$ is a pure bounded subgroup and so it has a complement L in A, which may be chosen to contain A_1. Choose a right transversal W of A_1 in L. Then the set $W\dot{B}$ is a right transversal of A_1 in A. With respect to this transversal, the triple (A, A_1, π_1) produces a transitive state-closed representation φ where $\dot{B}^\varphi = B$. By Proposition 3, we may rewrite A^φ as A. Then, the diagonal-topological closure A^* contains B^*. Let V be a complement of B in P. Each $\alpha \in A^*$ can be factored as $\alpha = \beta\gamma$ where $\beta \in B^*$ and γ is such that each of its states γ_u have activity $\sigma(\gamma_u) \in V$. Therefore, the set of these γ's is a group Γ such that $\Gamma = \Gamma^*$ and $A^* = \Gamma \oplus B^*$. Then, $(\Gamma, \Gamma \cap A_1, \pi_1)$ is a simple triple with $P(\Gamma)$ having exponent smaller than r. The proof is finished by induction on the exponent. □

The example below illustrates some of the ideas developed so far.

Example 3. *Let $m = 4, Y = \{1,2,3,4\}$and let σ be the cycle $(1,2,3,4)$. Furthermore, let $\alpha = (e,e,e,\alpha^2)\,\sigma \in \mathcal{A}(4)$ and let $A = \langle\alpha\rangle$. Then*

$$\begin{aligned} \alpha^2 &= (\alpha^2, e, e, \alpha^2)\,(1,3)\,(2,4)\,, \\ \alpha^4 &= \left(\alpha^2\right)^{(1)} = \alpha^{2x}, \left(\alpha^{2-x}\right)^2 = e. \end{aligned}$$

Then A is cyclic, torsion-free, transitive and state-closed, yet it is not diagonally closed, as $\alpha^x \notin A$. Even though A is torsion-free, its diagonal closure $\widetilde{A} = \left\langle \alpha^{x^i} \mid i \geq 0 \right\rangle$ is not; for $\kappa = \alpha^{2-x}$ has order 2. Let $K = \left\langle \kappa^{x^i} \mid i \geq 0 \right\rangle$. Then, $K \leq tor\left(\widetilde{A}\right)$ and it is direct to check that $\widetilde{A} = \langle\alpha, K\rangle$. Therefore, $K = tor\left(\widetilde{A}\right)$ and

$$\widetilde{A} = tor\left(\widetilde{A}\right) \oplus A.$$

Let $Y_1 = \{1,3\}, Y_2 = \{2,4\}$. Then $\{Y_1, Y_2\}$ is a complete block system for the action of α on Y. Also, α^2 induces the binary adding machine on both $\mathcal{T}(Y_1)$ and $\mathcal{T}(Y_2)$. The topological closure $\overline{A}$ of A is torsion-free and

$$\begin{aligned} tor(A^*) &= tor\left(\widetilde{A}\right), \\ A^* &= tor(A^*) \oplus \overline{A}. \end{aligned}$$

Moreover, $tor(A^*)$ induces a faithful state-closed, diagonally and topologically closed actions on the binary tree $\mathcal{T}(Y_1)$. Therefore, $tor(A^*)$ is isomorphic to $\frac{\mathbb{Z}}{2\mathbb{Z}}[[x]]$. Furthermore, α is represented as the binary adding machine on $\mathcal{T}(\{Y_1, Y_2\})$ and $\overline{A}$ is represented on this tree as the topological closure of the image of A.

6. Cyclic $\mathbb{Z}_m[[x]]$-modules

Cyclic automorphism groups $\langle\alpha\rangle$ of the tree, for which their state-diagonal-topological closure is isomorphic to a cyclic $\mathbb{Z}_m$-module have the form

$$\alpha = (\alpha^{q_1}, ..., \alpha^{q_m})\,\sigma$$

where $q_i \in \mathbb{Z}_m[[x]]$ for $1 \leq i \leq m$; here

$$\begin{aligned} q_i &= \sum_{j\geq 0} q_{ij} x^j, \\ q_{ij} &= \sum_{u\geq 0} q_{ij,u} m^u \in \mathbb{Z}_m. \end{aligned}$$

We prove

Theorem 8. *(i) The expression*

$$\alpha = (\alpha^{\mathbf{q_1}}, ..., \alpha^{\mathbf{q_m}})\,\sigma$$

is a well-defined automorphism of the m-ary tree. (ii) Let A be the state closure of $\langle\alpha\rangle$. Then $A^*$is abelian, isomorphic to the quotient ring $\frac{\mathbb{Z}_m[[x]]}{(r)}$ where

$$r = m - qx \text{ and } q = q_{\mathbf{1}} + ... + q_{\mathbf{m}}.$$

Proof. (1) Let $\sigma(l)$ denote the permutation induced by α on the l-th level. Then, the expression $\alpha = (\alpha^{\mathbf{q_1}}, ..., \alpha^{\mathbf{q_m}})\,\sigma$ represents

$$\begin{aligned} \sigma(1) &= \sigma, \\ \sigma(l) &= \left(\sigma(l-1)^{\overline{q_1}}, ..., \sigma(l-1)^{\overline{q_m}}\right)\sigma \end{aligned}$$

where $\overline{q_i} = \overline{q_{i0}} + \overline{q_{i1}}x + ... + \overline{q_{i(l-1)}}x^{l-1}$ and $\overline{q_{ij}} = q_{ij,0} + q_{ij,1}m + ... + q_{ij,l-1}m^{l-1}$.

(2.1) The states of α are words in α^p for $p \in \mathbb{Z}_m[[x]]$. Let $v = \alpha^{l_1}...\alpha^{l_a}, w = \alpha^{n_1}...\alpha^{n_b} \in A^*$. Then clearly $[v,w] \in Stab_A(1)$. We will prove that the entries of $[v,w]$ are products of conjugates of words in elements of the form $[\alpha^s, \alpha^t]$ where $s, t \in \mathbb{Z}_m[[x]]$.

Clearly $[v,w]$ can be developed into a word in conjugates of $\left[\alpha^{l_i}, \alpha^{n_j}\right]$.

Write $p=p_0+p'x, n=n_0+n'x$. We compute

$$\begin{aligned}\left[\alpha^p,\alpha^n\right] &= \left(\left[\alpha^{p_0},\alpha^{n'x}\right]\left[\alpha^{p_0},\alpha^{n_0}\right]^{\alpha^{n'x}}\right)^{\alpha^{p'x}} \\ &\quad \left[\alpha^{p'},\alpha^{n'}\right]^x\left[\alpha^{p'x},\alpha^{n_0}\right]^{\alpha^{n'x}} \\ &= \left[\alpha^{p_0},\alpha^{n'x}\right]^{\alpha^{p'x}}\left[\alpha^{p'},\alpha^{n'}\right]^x\left[\alpha^{p'x},\alpha^{n_0}\right]^{\alpha^{n'x}}.\end{aligned}$$

Therefore, we have to check $\left[\alpha^{\xi},\alpha^{nx}\right]$ where $\xi\in\mathbb{Z}_m, n\in\mathbb{Z}_m[[x]]$. Write $\xi=\xi_0+m\xi'$. Then,

$$\begin{aligned}\left[\alpha^{\xi},\alpha^{nx}\right] &= \left[\alpha^{\xi_0+m\xi'},\alpha^{nx}\right] \\ &= \left[\alpha^{\xi_0},\alpha^{nx}\right]^{\alpha^{m\xi'}}\left[\alpha^{m\xi'},\alpha^{nx}\right].\end{aligned}$$

Now,

$$\alpha^{\xi_0}=\left(v_{\mathbf{1}},v_{\mathbf{2}},...,v_{\mathbf{m}}\right)\sigma^{\xi_0},$$

where $v_{\mathbf{i}}$ are words in $\alpha^{q_{\mathbf{1}}},...,\alpha^{q_{\mathbf{m}}}$ and

$$\alpha^m=\left(\alpha^{q_{\mathbf{1}}}...\alpha^{q_{\mathbf{m}}},\alpha^{q_{\mathbf{2}}}...\alpha^{q_{\mathbf{m}}}\alpha^{q_{\mathbf{1}}},...,\alpha^{q_{\mathbf{m}}}\alpha^{q_{\mathbf{1}}}...\alpha^{q_{\mathbf{m-1}}}\right).$$

Therefore,

$$\left[\alpha^{\xi_0},\alpha^{nx}\right]=\left(\left[v_{\mathbf{1}},\alpha^n\right],...,\left[v_{\mathbf{m}},\alpha^n\right]\right)$$

and similarly,

$$\left[\alpha^{m\xi'},\alpha^{nx}\right]=\left(\left[\left(\alpha^{q_{\mathbf{1}}}...\alpha^{q_{\mathbf{m}}}\right)^{\xi'},\alpha^n\right],...,\left[\left(\alpha^{q_{\mathbf{m}}}\alpha^{q_{\mathbf{1}}}...\alpha^{q_{\mathbf{m-1}}}\right)^{\xi'},\alpha^n\right]\right).$$

Now we write $\beta=\alpha^{q_{\mathbf{1}}}...\alpha^{q_{\mathbf{m}}}$. Then $\left[\beta^{\xi'},\alpha^n\right]$ can be developed further as asserted. The same applies to the other entries.

(2.2) First, clearly $r\alpha=0$. Now let $u=u(x)$ annul α; write $u=u_0+u'x$ where $u_0=u(0)$. Then $m|u_0$ and so,

$$\begin{aligned}u &= m\frac{u_0}{m}+u'x=(xq)\frac{u_0}{m}+u'x+vr \\ &= xw_1+vr\end{aligned}$$

for some $v=v(x)$ and $w_1=q\frac{u_0}{m}+u'$. Then, xw_1 annuls α and so does w_1. On repeating, we find w_i such that $u\equiv x^iw_i$ modulo r and w_i annuls α for all $i\geq 1$.

In other words, $u\in\cap_{n\geq 1}(x\mathbb{Z})^n+(r)=(r)$. □

6.0.1. *The group* $D_m(j)$. Recall $\alpha=\left(e,..,e,\alpha^{x^{j-1}}\right)\sigma\in\mathcal{A}_m$. Then $\alpha^m=\alpha^{x^j}$; that is, $\alpha^r=e$ where $r=m-x^j$. The states of α are $\alpha,\alpha^x,...,\alpha^{x^{j-1}}$ and

$$D_m(j)=\left\langle\alpha,\alpha^x,...,\alpha^{x^{j-1}}\right\rangle;$$

therefore $D_m(j)$ is diagonally closed. The topological closure $\overline{D_m(j)}$ is isomorphic to the quotient ring $S=\frac{\mathbb{Z}_m[[x]]}{(r)}$ which is clearly a free $\mathbb{Z}_m$-module of rank j.

6.1. **The case $P(A)$ cyclic of prime order.**

Theorem 9. *Let m be a prime number. Let A be a torsion-free abelian transitive state-closed subgroup of $\mathcal{A}_m$. Let $\beta \in A\backslash Stab_A(j)$. Then $A^* = \langle\beta\rangle^*$ and is topologically finitely generated. Furthermore, A^* is conjugate to $\overline{D_m(j)}$ for some $j \geq 1$.*

The proof is developed in four steps.

Step 1. For $z \in A$, define $\zeta(z) = j$ such that $z^m \in Stab(j)\backslash Stab(j+1)$. As A is torsion-free, $\zeta(z)$ is finite for all nontrivial z and $z^m = (v)^{(j)}, v \in A\backslash Stab_A(1)$.

Choose $\beta = (\beta_1, \beta_2, ..., \beta_m)\sigma \in A\backslash Stab_A(1)$ having minimum $\zeta(\beta) = j$. If $z \in Stab_A(1), z \neq e$, then there exists $l > 0$ such that $z^m = (c)^{(l)}$ and $c \in A\backslash Stab_A(1)$. Therefore, by minimality of β we have $\zeta(c) \geq \zeta(\beta)$ and $\zeta(z) > \zeta(\beta)$.

Lemma 5. *(Uniform gap) Let $z \in Stab_A(1)$. Then $\zeta(z\beta) = \zeta(\beta)$.*

Proof. First note that

$$\begin{aligned}\beta^m &= (\beta_1\beta_2...\beta_m)^{(1)},\\ \beta_1\beta_2...\beta_m &= (\gamma)^{(j-1)}, \gamma \in A\backslash Stab_A(1).\end{aligned}$$

We have $z = c^{(1)}$ and $z\beta = (c\beta_1, c\beta_2, ..., c\beta_m)\sigma, (z\beta)^m = (u)^{(1)}$ where $u = c^m\beta_1...\beta_m = c^m(\gamma)^{(j-1)}$. If $c \in A\backslash Stab_A(1)$ then $\zeta(c) = n \geq j, c^m \in Stab(n)\backslash Stab(n+1)$ and so, $u \in Stab_A(j-1)\backslash Stab_A(j)$. If $c \in Stab_A(1)$ then $\zeta(c) > j$ and so, $c^m \in Stab(k)$ where $k > j$ and again $u \in Stab(j-1)\backslash Stab(j)$. $\square$

Step 2. Note that

$$\begin{aligned}\beta^m &= (\gamma)^{(j)}, \quad \gamma^m = (\lambda)^{(j)}\\ \beta^{m^2} &= (\lambda)^{(2j)}\end{aligned}$$

where by the uniform gap lemma above, $\gamma, \lambda \in A\backslash Stab_A(1)$. Therefore, on repeating this process, we find that β^{m^s} induces $\sigma^{(sj)}$ on the (sj)th level of the tree for all $s \geq 0$. Now given a level $t \geq 0$, on dividing t by j, we get $t = sj + i$ with $0 \leq i \leq j-1$ and then $\left(\beta^{(i)}\right)^{m^s} = \left(\beta^{m^s}\right)^{(i)}$ induces $\left(\sigma^{(sj)}\right)^{(i)} = \sigma^{(sj+i)} = \sigma^{(t)}$ on the t-th level of the tree. It follows that the group A is a subgroup of the topological closure of $\left\langle \beta, \beta^{(1)}, ..., \beta^{(j-1)} \right\rangle$.

Step 3. We have for $\beta = (\beta_1, \beta_2, ..., \beta_m)\sigma$,

$$\beta_i = \beta^{p_i}, \quad p_i = r_{i0} + r_{i1}x + ... + r_{i(j-1)}x^{j-1} \in \mathbb{Z}_m[x],$$

and

$$\begin{aligned}\beta^m &= (\beta_1\beta_2...\beta_m)^{(1)},\\ \beta_1\beta_2...\beta_m &= \beta^{p_1+...+p_m},\\ p_1 + ... + p_m &= q.x^{j-1}\end{aligned}$$

where q is an invertible element of $\mathbb{Z}_m[[x]]$.

Proposition 5. *The element $\beta = (\beta_1, \beta_2, ..., \beta_m)\sigma$ is conjugate in $\mathcal{A}_m$ to $\alpha = (e, ..., e, \alpha^{x^{j-1}})\sigma$.*

Proof. Let $h = (h_1, h_2, ..., h_m)$ be an automorphism of the tree. Then

$$\beta^h = (h_1^{-1}\beta_1 h_2, h_2^{-1}\beta_2 h_3, ..., h_m^{-1}\beta_m h_1)\sigma.$$

Therefore $\beta^h = \alpha$ holds if and only if

$$h_2 = \beta_1^{-1}h_1,\ h_3 = \beta_2^{-1}h_2, ...,\ h_m = \beta_{m-1}^{-1}h_{m-1},\ h_1 = \beta_m^{-1}h_m\alpha^{x^{j-1}}.$$

These conditions can be rewritten as

$$\begin{aligned} h_2 &= \beta_1^{-1}h_1,\ h_3 = \beta_2^{-1}\beta_1^{-1}h_1, ...,\ h_m = \beta_{m-1}^{-1}...\beta_1^{-1}h_1, \\ h_1 &= \beta_m^{-1}\beta_{m-1}^{-1}...\beta_1^{-1}h_1\alpha^{x^{j-1}}, \end{aligned}$$

or as

$$\begin{aligned} h &= (h_1, \beta_1^{-1}h_1, \beta_2^{-1}\beta_1^{-1}h_1..., \beta_{m-1}^{-1}...\beta_1^{-1}h_1) \\ &= (e,\ \beta_1^{-1},\ \beta_2^{-1}\beta_1^{-1}...,\ \beta_{m-1}^{-1}...\beta_1^{-1})\,(h_1)^{(1)}, \end{aligned}$$

and

$$(\beta_1\beta_2...\beta_m)^{h_1} = \alpha^{x^{j-1}}.$$

Since

$$\beta_1\beta_2...\beta_m = \beta^{q.x^{j-1}},$$

we repeat the above procedure replacing β by β^q and replacing h_1 by $(h_1')^{x^{j-1}}$. This leads to the conjugation equation

$$(\beta^q)^{h_1'} = \alpha.$$

In this manner, we determine an automorphism h of the tree which effects the required conjugation

$$\beta^h = \alpha.$$

□

Example 4. *Let $\beta = (e, \beta^q)\sigma$ where $q = 1 + x$. Then β is conjugate to the adding machine $\alpha = (e, \alpha)\sigma$. Note that from Example 1, β is not obtainable from α by simply choosing a different transversal. To exhibit the conjugator $h : \beta \to \alpha$, constructed in the proof, define the polynomial sequences*

$$\begin{aligned} c_0 &= 1,\ \ c_1 = q,\ \ c_n = 2c_{n-2} + c_{n-1}; \\ c_{-1}' &= 0,\ \ c_0' = 0,\ \ c_n' = c_{n-1} + c_{n-1}'. \end{aligned}$$

Then

$$h = (e, e)^{(0)}\left(e, \beta^{-1}\right)^{(1)}\left(e, \beta^{-(1+q)}\right)^{(2)} ... \left(e, \beta^{-c_n'}\right)^{(n)} ....$$

Step 4. By Proposition 2, we have $A \leq \overline{A} = C_{\mathcal{A}}(\alpha)$ and

$$A^h \leq C_{\mathcal{A}}\left(\alpha^h\right) = C_{\mathcal{A}}(\beta) = \overline{D_m(j)}.$$

This finishes the proof of the theorem.

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